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Vibration analysis of a circular arch with variable cross-section using differential transformation and generalized differential quadrature

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Abstract

Vibration analysis of circular arches is an important subject in mechanics due to its various applications. In particular, circular arches with variable cross-section have been widely used to satisfy modern architectural and structural requirements. Recently, the generalized differential quadrature method (GDQM) and differential transformation method (DTM) were proposed by Shu and Zhou, respectively. In this study, GDQM and DTM are applied to vibration analysis of circular arches with variable cross-section. The governing equation of motion is derived and the non-dimensional natural frequencies are obtained for various boundary conditions. The concepts of differential transformation and generalized differential quadrature are briefly introduced. The results obtained by these methods are compared with previously published works. GDQM and DTM showed fast convergence, accuracy and validity in solving the vibration problem for circular arches with variable cross-sections.

1. Introduction

Arches are widely used as basic structural elements. As such, analysis of the vibration of an arch structure is essential in civil engineering, architecture, marine engineering and aeronautics. In particular, the free vibration, which is a function of the natural properties of structures, is an important subject of investigation.

Generally, much research has been conducted using the Bernoulli–Euler beam theory to analyze free vibration of members with uniform cross-section and circular arches with varying cross-section, where the arch's neutral axis is inextensible [1–6].

Laura and Verniere de Irassar studied the vibration of arch structures with linearly varying cross-section and with end mass [1]. Gutierrez and Laura carried out vibration analysis of arches with different forms of varying cross-section using the Ritz method [2]. Auciello and Rosa analyzed and investigated the vibration of

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circular arches with varying cross-section using various numerical methods, such as the cell discretization method (CDM), finite element method (FEM), Galerkin method, and Ritz method [3]. Tong et al. obtained an analytical solution by discretizing the varying cross-section of a circular arch into finite elements.

Recent research using FEM, Ritz, CDM and Galerkin numerical methods involved complicated equations that were tedious to calculate. The differential transformation method (DTM) and generalized differential quadrature method (GDQM) have been developed as a novel approach in obtaining rapid convergence with relatively straightforward equations.

The objective of this study is to analyze the free vibration of circular arches with varying cross-section using DTM and GDQM.

In this study, the concepts of differential transformation and generalized differential quadrature are briefly introduced. The governing equations of motion are derived and the vibration analysis is accomplished using GDQM and DTM. The non-dimensional natural frequencies were obtained for various boundary conditions and the results are compared with that of previous works using conventional methods. Results showed that GDQM and DTM exhibited rapid convergence, accuracy and validity in solving the vibration problem of circular arches with variable cross-section.

2. Governing equation

Fig. 1 shows a circular arch with a varying cross-section and opening angle θ at an arbitrary point, where R denotes the circular arch radius, I_0 is the moment of inertia, A_0 is the area at the top of arch, and w_r, v_r and ϕ represent the tangential displacement, radial displacement and rotation angle at the arbitrary point, respectively.

Fig. 2 shows the loads acting on a circular arch element in free vibration, where M, N and T denote the bending moment, axial force and shear force, respectively. The governing equation of motion for a circular arch with varying cross-section with respect to small displacement w is derived, shown in Eq. (1). The shear and axial forces acting on the small element are represented by Eqs. (2) and (3), respectively:

$$\frac{\partial^3}{\partial\theta^3}\left[E\bar{I}(\theta)\left(\frac{\partial^3 w}{\partial\theta^3}+\frac{\partial w}{\partial\theta}\right)\right]+\frac{\partial}{\partial\theta}\left[E\bar{I}(\theta)\left(\frac{\partial^3 w}{\partial\theta^3}+\frac{\partial w}{\partial\theta}\right)\right]+R^4\frac{\partial}{\partial\theta}\left[\rho\bar{A}(\theta)\frac{\partial^3 w}{\partial\theta\partial t^2}\right]-\rho\bar{A}(\theta)R^4\frac{\partial^2 w}{\partial t^2}=0, \tag{1}$$

$$T(\theta,t)=-\frac{1}{R^3}\frac{\partial}{\partial\theta}\left[E\bar{I}(\theta)\left(\frac{\partial^3 w}{\partial\theta^3}+\frac{\partial w}{\partial\theta}\right)\right], \tag{2}$$

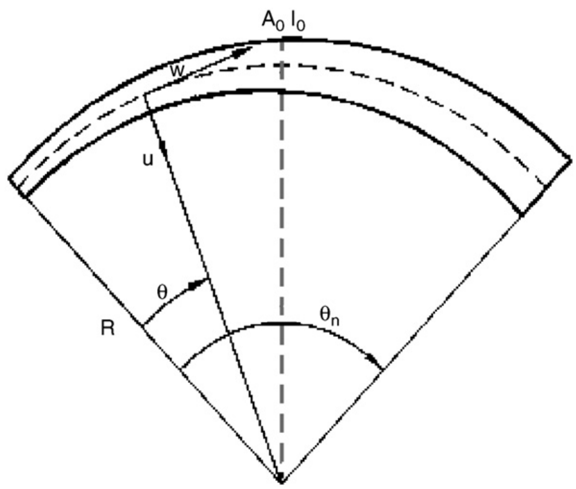

Fig. 1. A circular arch with variable cross-section.

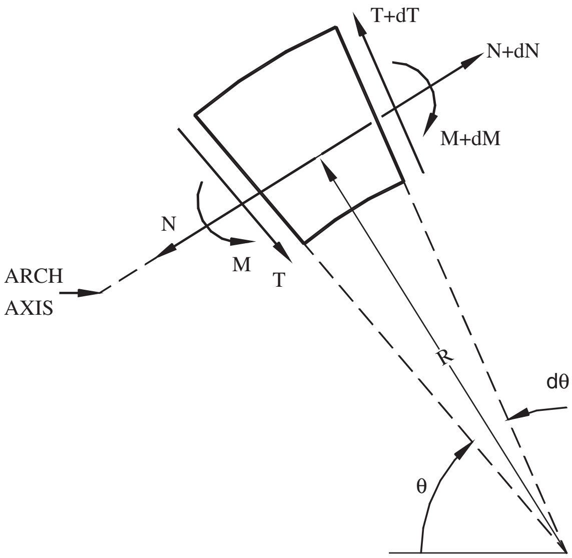

Fig. 2. Loads on a circular arch element.

$$N(\theta,t) = -\frac{\partial T(\theta,t)}{\partial\theta} + \rho\bar{A}(\theta)R\frac{\partial^2 u(\theta,t)}{\partial t^2} = -\frac{1}{R^3}\frac{\partial^2}{\partial\theta^2}\left[E\bar{I}(\theta)\left(\frac{\partial^3 w}{\partial\theta^3}+\frac{\partial w}{\partial\theta}\right)\right] + \rho\bar{A}(\theta)R\frac{\partial^3 w}{\partial\theta\partial t^2}. \tag{3}$$

The cross-section used in the above governing equation is considered to have constant width and variable height. If the width and height of the cross-section at the top of arch are b and h, respectively, the area is $A_0 = bh$ and the moment of inertia is $I_0 = bh^3/12$. Using the function f with respect to the opening angle θ, the variation of the cross-sectional area $\bar{A}(\theta)$ and the moment of inertia $\bar{I}(\theta)$ with respect to arbitrary θ are represented by

$$\bar{A}(\theta) = A_0 f = A_0\sum_{i=0}^{r} f_i\theta^i, \quad \bar{I}(\theta) = I_0 q = I_0\sum_{i=0}^{p} q_i\theta^i, \tag{4}$$

where r and p are the coefficients of the polynomial expressions for $\bar{A}(\theta)$ and $\bar{I}(\theta)$, respectively.

Eq. (4) can be applied to an arbitrary varying cross-section, and the relationship between the arch cross-section function f and the moment of inertia q is $q = f^3$.

If the vibration of the circular arch with varying cross-section is assumed to be harmonic, then

$$w(\theta,t) = W(\theta)e^{i\omega t}. \tag{5}$$

Substituting Eq. (5) into Eq. (1) and eliminating $e^{i\omega t}$, the equation of motion for a circular arch with varying cross-section becomes

$$\frac{d^3}{d\theta^3}\left[q\left(\frac{d^3W}{d\theta^3}+\frac{dW}{d\theta}\right)\right] + \frac{d}{d\theta}\left[q\left(\frac{d^3W}{d\theta^3}+\frac{dW}{d\theta}\right)\right] + \frac{d}{d\theta}\left[-\frac{\rho A_0 R^4}{EI_0}\omega^2 f\frac{dW}{d\theta}\right] + \frac{\rho A_0 R^4}{EI_0}\omega^2 fW = 0, \tag{6}$$

where ω is the natural angular frequency (rad/s) of vibration.

The variable $X = \theta/\theta_n$ is introduced to present the governing equation in non-dimensional form, where θ_n is the opening angle of the arch.

Eq. (6) becomes

$$\frac{d^3}{\theta_n^3 dX^3}\left[q(X)\left(\frac{d^3W}{\theta_n^3 dX^3}+\frac{dW}{\theta_n dX}\right)\right] + \frac{d}{\theta_n dX}\left[q(X)\left(\frac{d^3W}{\theta_n^3 dX^3}+\frac{dW}{\theta_n dX}\right)\right] + \frac{d}{\theta_n dX}\left[-\lambda^2 f(X)\frac{dW}{\theta_n dX}\right] + \lambda^2 f(X)W = 0, \tag{7}$$

where $\lambda = \sqrt{(\rho A_0 R^4/EI_0)}\omega$ is the non-dimensional natural frequency.

In this study, the boundary conditions considered are clamped–clamped and hinged–hinged.
The boundary conditions for clamped and hinged edges are as follows:

$$\text{clamped}: w = 0,\ w' = 0,\ \psi = 0 \quad \text{at } X = 0,\ 1, \tag{8}$$

$$\text{hinged}: w = 0,\ w' = 0,\ M = 0 \quad \text{at } X = 0,\ 1. \tag{9}$$

The mathematical expressions for both the rotation angle of the cross-section ψ and the bending moment M are

$$\psi = \frac{1}{R}\left(\frac{\mathrm{d}^2 w}{\theta_n^2\,\mathrm{d}X^2} + w\right), \tag{10}$$

$$M = -\frac{EI_0 q}{R^2}\left(\frac{\mathrm{d}^3 w}{\theta_n^3\,\mathrm{d}X^3} + \frac{\mathrm{d}w}{\theta\,\mathrm{d}X}\right). \tag{11}$$

3. Differential transformation method and generalized differential quadrature method

3.1. Differential transformation method

The DTM is based on the Taylor series expansion. Solution of the equations is obtained through recursive algebra of the transformed governing equations of motion by basic mathematical operations. DTM is a very useful method for solving linear and nonlinear problems.

Differential transformation of an arbitrary original function is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{\mathrm{d}^k y(x)}{\mathrm{d}x^k}\right]_{x=0}, \tag{12}$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function, which is called the T-function.

The differential inverse transformation of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k). \tag{13}$$

Substituting Eq. (12) into Eq. (13) and rearranging, the original function $y(x)$ can be expressed as

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{\mathrm{d}^k y(x)}{\mathrm{d}x^k}\right]_{x=0}. \tag{14}$$

In real applications, $y(x)$ can be approximated using finite terms and Eq. (13) can be written as

$$y(x) = \sum_{k=0}^{n} x^k Y(k), \tag{15}$$

where n is a natural number determined by the convergence of the solution. Eq. (15) implies that $y(x) = \sum_{k=n+1}^{\infty} x^k Y(k)$ is negligibly small.

Examples of differential transformation of the original function are listed in Table 1.

3.2. Generalized differential quadrature method

The GDQM has recently been proposed as a general numerical method for solving high-order ordinary and partial differential equations. GDQM is a generalization of the conventional differential quadrature method, since it can be applied to any finite-order differential equation in a strict form. GDQM uses the same number of independent variables as for the conditions at a point [12–15].

Table 1
Examples of differential transformation of the original function

Original function	T-function
$w(x)=y(x)\pm z(x)$	$W(k)=Y(k)\pm Z(k)$
$z(x)=\lambda y(x)$	$Z(k)=\lambda Y(k)$
$w(x)=\frac{d^n y(x)}{dx^n}$	$W(k)=(k+1)(k+2)\cdots(k+n)Y(k+n)$
$w(x)=y(x)z(x)$	$W(k)=\sum_{l=0}^{k}Y(l)Z(k-l)$
$w(x)=x^m$	$W(k)=\delta(k-m)=\begin{cases}1 & \text{for } k=m\\ 0 & k\neq m\end{cases}$
$w(x)=\sin(\lambda x)$	$W(k)=\frac{\lambda^k}{k!}\sin\left(\frac{\pi k}{2}\right)$

According to the concept of conventional integral quadrature, the nth-order derivative with respect to x of the function $u(x, t)$ at the ith grid point was approximated using Eq. (16) by Bellman et al. [13]:

$$\frac{\partial^n}{\partial x^n}[u_x(x_i,t)]=\sum_{j=1}^{N}c_{ij}^{(n)}u(x_j,t)\quad \text{for } i=1,\ 2,\dots,N, \tag{16}$$

where $u_x(x_i,t)$ indicates the nth-order derivative of $u(x, t)$ with respect to x at x_i, N is the number of discrete grid points, and $c_{ij}^{(n)}$ are the weighting coefficients.

Weighting coefficients with respect to the first-order derivative in the GDQM can be represented by the following equations:

$$c_{ij}^{(1)}=\frac{M^{(1)}(x_i)}{(x_i-x_j)M^{(1)}(x_j)},\quad i,j=1,\ 2,\dots,N \ \text{ but } i\neq j, \tag{17}$$

$$c_{ii}^{(1)}=-\sum_{j=1,j\neq i}^{N}c_{ij}^{(1)},\ i=1,\ 2,\dots,N\quad \text{for } i=j, \tag{18}$$

where

$$M^{(1)}(x_i)=\sum_{j=1,j\neq i}^{N}(x_i-x_j).$$

Weighting coefficients for second-order or higher-order derivatives can be obtained from the recurrence relationship of the mth-order weighting coefficients $c_{ij}^{(m)}$ at $u_x^{(m)}(x_i,t)$ represented in Eqs. (19) and (20):

$$c_{ij}^{(m)}=m\left(c_{ii}^{(m-1)}c_{ij}^{(1)}-\frac{c_{ij}^{(m-1)}}{x_i-x_j}\right)\quad \text{for } i\neq j,\ m=2,3,\dots,N-1;\ i,j=1,2,\dots,N, \tag{19}$$

$$c_{ii}^{(m)}=-\sum_{j=1,j\neq i}^{N}c_{ij}^{(m)}\quad \text{for } i=1,2,\dots,N. \tag{20}$$

As illustrated above, the equations to determine weighting coefficients for GDQM are relatively more concise, straightforward and less cumbersome to formulate and program by recursive relationships.

4. Numerical results and discussion

4.1. Differential transformation method

Numerical DTM analysis is based on the following matrix equation composed of the transformed differential equations of motion and the transformed boundary condition equations:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & a_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} & a_{n+1,n+1} \end{bmatrix} \begin{bmatrix} U(0) \\ U(1) \\ \vdots \\ U(n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (21)$$

For a non-trivial solution, the determinant of the coefficient matrix of the above equation must vanish, that is

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & a_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} & a_{n+1,n+1} \end{vmatrix} = 0. \quad (22)$$

The natural frequencies of a circular arch can be obtained from Eq. (22).

4.2. Generalized differential quadrature method

Numerical GDQM analysis is represented by a matrix divided by the interior and boundary portions from the transformed equation of motion [12]:

$$[A_{IB}]\{W_B\} + [A_{II}]\{W_I\} = \lambda^2\{W_I\}, \quad (23)$$

where $[A_{IB}]$ and $[A_{II}]$ are coefficient matrices of the boundary and interior portions of the governing equation, respectively.

The transformed boundary condition equation is represented in the following matrix form:

$$[A_{BB}]\{W_B\} + [A_{BI}]\{W_I\} = 0, \quad (24)$$

where $[A_{BB}]$ and $[A_{BI}]$ are coefficient matrices of the boundary and interior portions of the boundary condition equations, respectively.

Substituting Eq. (24) into Eq. (23) and rearranging, the vibration analysis of an arch is represented by the following eigenvalue problem:

$$\left([A_{II}] - [A_{IB}][A_{BB}]^{-1}[A_{BI}]\right)\{W_I\} = \lambda^2\{W_I\}. \quad (25)$$

In this numerical analysis, a Chebyshev polynomial grid is used in the grid distribution, as represented by Eq. (26):

$$X_i = \frac{1}{2}\left(1 - \cos\frac{\pi(i-1)}{(N-1)}\right), \quad i = 1, 2, 3, \ldots, N. \quad (26)$$

4.3. Circular arches with continuously varying cross-section

4.3.1. Continuously varying cross-section

If a circular arch has a cross-section of constant width and linearly varying height, the function for the varying cross-section can be expressed as follows:

$$f(X) = 1 + \alpha(2X - 1) \quad \text{for } 0 \leqslant X \leqslant 1, \quad (27)$$

where α is the area parameter.

The numerical analysis of a circular arch with linearly varying cross-section is applied to three conditions: clamped–clamped, hinged–hinged and clamped–hinged. The natural frequencies are obtained to five significant figures and compared to those in Ref. [3].

Tables 2 and 3 show numerical results for $\alpha = 0.1$ for the boundary conditions clamped–clamped and hinged–hinged. The non-dimensional natural frequencies obtained by DTM and GDQM show good agreement, but are always greater than the non-dimensional natural frequencies calculated by the Rayleigh–Ritz method and CDM. For this reason, Rayleigh–Ritz method and CDM always have lower limit values [3]. However, the results of FEM and those for DTM and GDQM agree comparatively well.

Fig. 3 shows the convergence of the non-dimensional natural frequencies obtained by DTM and GDQM. The GDQM solutions show a converging trend starting from a relatively lower grid number 13 then simultaneously converge at 18 for the first and second non-dimensional natural frequencies.

The DTM solutions show a converging trend starting from series term 28 and are completely converged at 38. The second non-dimensional natural frequency shows a converging trend from 37 and is converged at 47. The DTM results show that the first and second natural frequencies are sequentially obtained and that higher natural frequencies converge with an increasing number of series terms. In contrast, for GDQM the first and second natural frequencies are simultaneously obtained.

4.3.2. Varying cross-section with quadratic function

If the width of a circular arch's cross-section is constant and its height is a quadratic variable with respect to X, the function of the varying cross-section is represented as follows:

$$f(X) = (1 + \alpha(2X - 1))^2 \quad \text{for } 0 \leqslant X \leqslant 1. \tag{28}$$

Tables 4 and 5 show the non-dimensional natural frequencies of a quadratically varying cross-section for $\alpha = 0.1$ for the boundary conditions clamped–clamped and hinged–hinged. The results for the first and second non-dimensional natural frequencies by DTM and GDQM show good agreement.

Table 2
Non-dimensional fundamental frequencies of clamped-clamped circular arches with continuously varying cross-section: $= 1 + \alpha(2X - 1)$

θ_n (deg)	Method				
	Rayleigh–Ritz (3)	FEM (3)	CDM (3)	DTM	GDQM
$\alpha = 0.1$					
10	1999.9		2000.5	2017.0	2017.0
20	498.33	502.52	499.44	502.30	502.30
30	220.06		220.56	221.82	221.82
40	122.70	123.65	122.97	123.67	123.67
50	77.632		77.813	78.258	78.258
60	53.172	53.674	53.303	53.607	53.607

Table 3
Non-dimensional fundamental frequencies of hinged–hinged circular arches with continuously varying cross-section: $f(X) = 1 + \alpha(2X - 1)$

θ_n (deg)	Method				
	Rayleigh–Ritz (3)	FEM (3)	CDM (3)	DTM	GDQM
$\alpha = 0.1$					
10	1286.5		1287.8	1290.5	1290.5
20	319.09	320.81	320.11	320.76	320.76
30	141.05		140.92	141.20	141.20
40	78.069	78.438	78.220	78.373	78.373
50	49.124		49.218	49.312	49.312
60	33.430	33.621	33.484	33.546	33.546

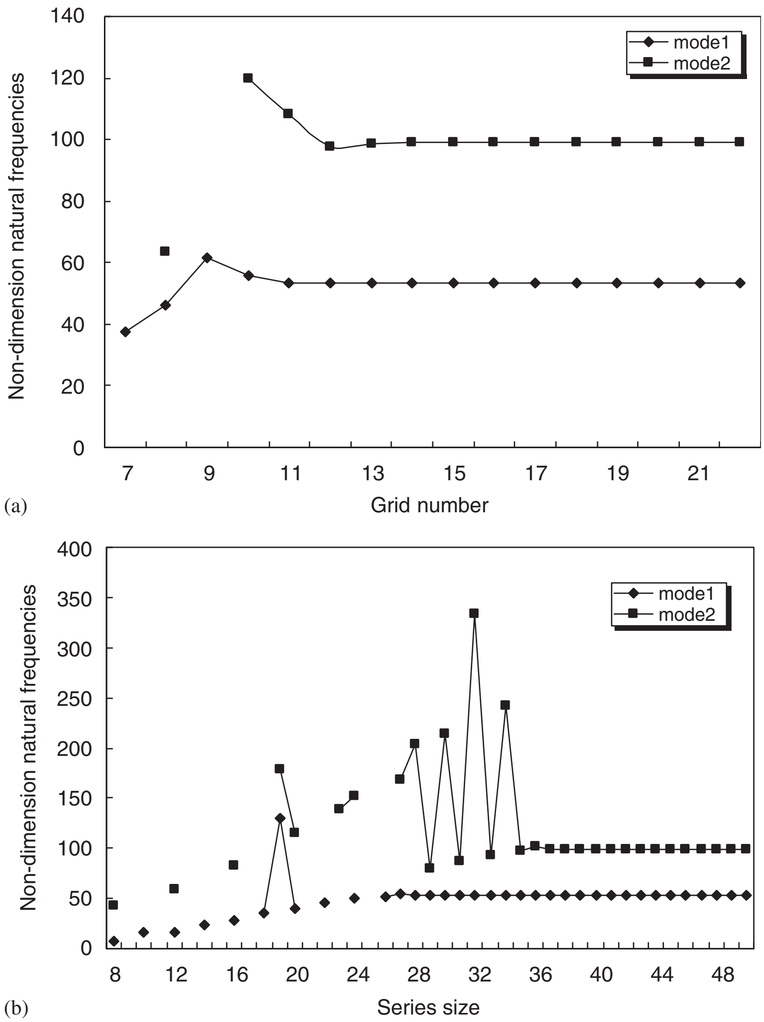

Fig. 3. Convergence of non-dimensional natural frequencies of a clamped–clamped circular arch with continuously varying cross-section: ($\theta = 60^\circ$ and $f(X) = 1 + 0.1(2X - 1)$): (a) GDQ solution and (b) DT solution.

Table 4
Non-dimensional natural frequencies of clamped–clamped circular arches with continuously varying cross-section

θ_n (deg)	Method			
	DTM		GDQM	
	Mode 1	Mode 2	Mode 1	Mode 2
$\alpha = 0.1$				
10	2012.2	3622.0	2012.2	3622.0
20	501.12	904.18	501.12	904.18
30	221.30	400.88	221.30	400.88
40	123.39	224.73	123.39	224.73
50	78.082	143.20	78.082	143.20
60	53.491	98.920	53.491	98.920

Table 5
Non-dimensional natural frequencies of hinged–hinged circular arches with continuously varying cross-section

θ_n (deg)	Method			
	DTM		GDQM	
	Mode 1	Mode 2	Mode 1	Mode 2
$\alpha = 0.1$				
10	1285.1	2750.4	1285.1	2750.4
20	319.43	686.25	319.43	686.25
30	140.61	304.00	140.61	304.00
40	78.046	170.22	78.046	170.22
50	49.106	108.30	49.106	108.30
60	33.405	74.671	33.405	74.671

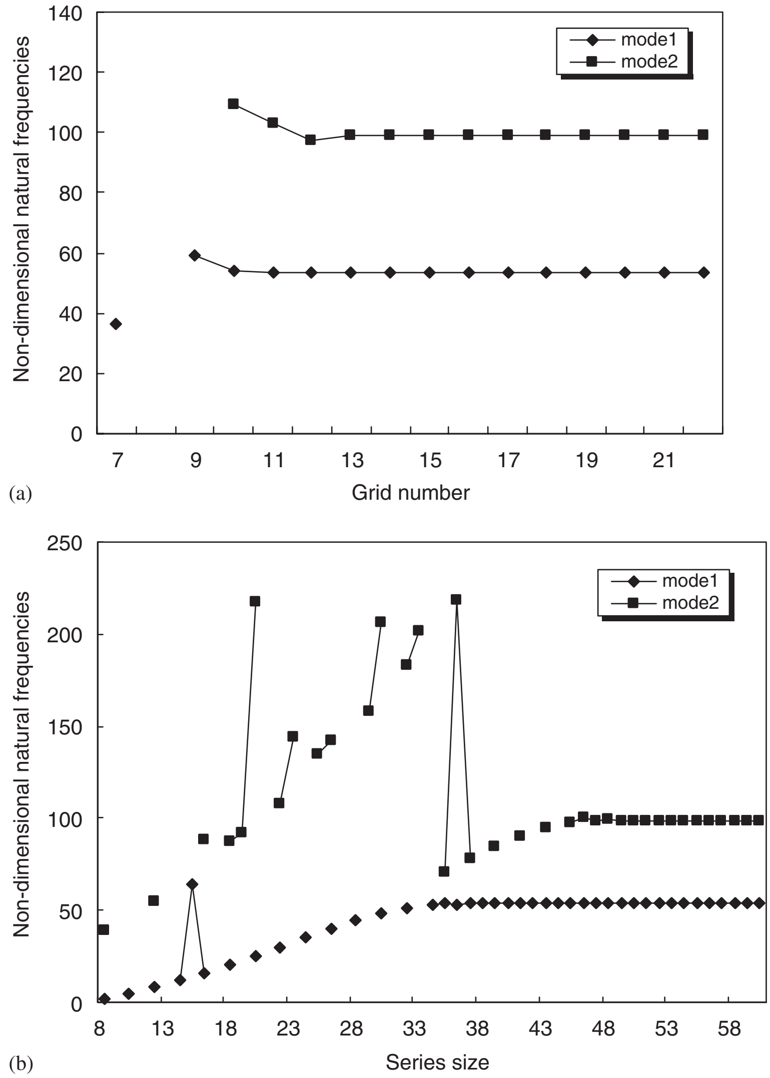

Fig. 4. Convergence of non-dimensional natural frequencies of a clamped–clamped circular arch with continuously varying cross-section ($\theta = 60^\circ$ and $f(X) = (1 + 0.1(2X - 1))^2$): (a) GDQM solution and (b) DT solution.

Fig. 4 show the convergence of the non-dimensional natural frequencies calculated using both DTM and GDQM. The numerical GDQM results indicate that the first and second non-dimensional natural frequencies show a converging trend from grid number 10 and are converged at grid number 18. The first non-dimensional natural frequency by DTM shows a converging trend from series term 30 and is completely converged at 38. The second non-dimensional natural frequency shows a converging trend from series term 40 and is converged at 52.

The results in Figs. 3 and 4 show that DTM is influenced by the coefficient of the cross-section function. Also the convergence number of the non-dimensional natural frequency increases with increasing function coefficient. In contrast, the coefficient of the GDQM cross-section function does not influence the convergence. The first and second non-dimensional natural frequencies of GDQM simultaneously converge at a specific grid number.

4.3.3. Varying cross-section with sine function

If the width is constant and the height varies as a sine function for the cross-section of a circular arch, the function of the varying cross-section is represented by the following equation:

$$f(X) = 1 - \alpha(\sin(\pi X - 1)). \tag{29}$$

Tables 6 and 7 show the non-dimensional natural frequencies for a circular arch of varying cross-section represented by the sine function, for clamped–clamped and hinged–hinged boundary conditions in the case of $\alpha = 0.1$.

Table 6
Non-dimensional natural frequencies of clamped–clamped circular arches with continuously varying cross-section: $f(X) = 1 - \alpha(\sin(\pi X - 1))$

θ_n (deg)	Method			
	DTM		GDQM	
	Mode 1	Mode 2	Mode 1	Mode 2
$\alpha = 0.1$				
10	2127.1	3816.5	2127.1	3816.5
20	529.82	952.76	529.82	952.76
30	234.04	422.45	234.04	422.45
40	130.53	236.84	130.53	236.84
50	82.635	150.94	82.635	150.94
60	56.638	104.28	56.638	104.28

Table 7
Non-dimensional natural frequencies of hinged–hinged circular arches with continuously varying cross-section: $f(X) = 1 - \alpha(\sin(\pi X - 1))$

θ_n (deg)	Method			
	DTM		GDQM	
	Mode 1	Mode 2	Mode 1	Mode 2
$\alpha = 0.1$				
10	1333.8	2874.8	1333.8	2874.8
20	331.58	717.30	331.58	717.30
30	145.99	317.77	145.99	317.77
40	81.050	177.94	81.050	177.94
50	51.012	113.22	51.012	113.22
60	34.715	78.067	34.715	78.067

The natural frequency results for cross-sections with a sine function are greater than those for constant cross-sections. In addition, the results of DTM and GDQM exhibit similar trends and are in good agreement.

5. Conclusions

In this paper, the vibration of a circular arch with variable cross-section and various boundary conditions was analyzed using DTM and GDQM. The results were compared with those obtained using conventional methods (FEM, Ritz method, CDM) in order to validate and verify the numerical analysis employed.

From these, the following conclusions can be stated:

1. The results obtained using DTM and GDQM showed good agreement.
2. Using DTM, accurate and efficient results were obtained as the series term increased.
3. GDQM obtained accurate results in spite of using a relatively lower grid number of 20.
4. In the convergence of non-dimensional natural frequency, DTM showed sensitivity to the coefficient of the varying cross-section function while GDQM did not.
5. DTM and GDQM showed rapid convergence, accuracy and validity in solving the vibration problem for circular arches with variable cross-section.

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